

FURSTENBERG ENTROPY VALUES FOR NONSINGULAR ACTIONS OF GROUPS WITHOUT PROPERTY (T)

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ABSTRACT. Let G be a discrete countable infinite group that does not have Kazhdan's property (T) and let κ be a generating probability measure on G . Then for each $t > 0$, there is a type III_1 ergodic free nonsingular G -action whose κ -entropy (or the Furstenberg entropy) is t .

0. INTRODUCTION

Let G be a discrete countable infinite group. A probability measure κ on G is called *generating* if the support of κ generates G as a semigroup. Let $T = (T_g)_{g \in G}$ be a nonsingular action of G on a standard probability space (X, \mathfrak{B}, μ) . The *Furstenberg entropy* (or κ -entropy) of T is defined by

$$h_\kappa(T, \mu) := - \sum_{g \in G} \kappa(g) \int_X \log \frac{d\mu \circ T_g}{d\mu}(x) d\mu(x)$$

(see [Fu]). Jensen's inequality implies that $h_\kappa(T, \mu) \geq 0$ and that (for generating measures) equality holds if and only if μ is invariant under T . Of course, the κ -entropy is invariant under conjugacy. If $\sum_{g \in G} \kappa(g) \frac{d\mu \circ T_g}{d\mu}(x) = 1$ for a.e. $x \in X$ then T is called κ -stationary. *Furstenberg entropy realization problem* is to describe all values that κ -entropy takes on the set of κ -stationary actions. The problem appears quite difficult. Some progress was achieved in recent papers [NeZi], [Ne], [Bo], [HaTa]. To state one of the results on the entropy realization problem we first recall that G has Kazhdan's *property (T)* if every unitary representation of G which has almost invariant vectors admits a nonzero invariant vector (see [Be–Va]). It was shown in [Ne] that if G has property (T) then for every generating measure κ , the pair (G, κ) has an *entropy gap*, i.e. there exists some constant $\epsilon = \epsilon(G, \kappa) > 0$ such that the $h_\kappa(T) > \epsilon$ for each purely infinite ergodic stationary G -action T . We recall that an ergodic action is called *purely infinite* if it does not admit an equivalent invariant probability measure. In [Bo–Ta] the converse statement was proved: if T does not have property (T) then for each generating measure κ ,

$$(0-1) \quad \inf \{h_\kappa(T, \mu) \mid T \text{ is purely infinite, ergodic, } \mu\text{-nonsingular action of } G\} = 0.$$

We note that the authors of [Bo–Ta] consider κ -entropy values on arbitrary (not only stationary as in the other aforementioned papers) purely infinite nonsingular actions. They also show that the entropy gap for (G, κ) established in [Ne] for the stationary actions holds also for all (purely infinite) ergodic nonsingular actions.

In this connection we note that if a purely infinite ergodic G -action is stationary then the space of the action is non-atomic. However, in the general (non-stationary) case considered in [Bo–Ta], there exist purely infinite transitive G -actions on purely atomic measure spaces. In particular, the action of G on itself via rotations is free, nonconservative, ergodic and purely infinite. We consider such actions as *pathological*. Unfortunately, the proof the main result from [Bo–Ta] does not exclude appearance of pathological actions in (0-1).

Our purpose in the present paper is to refine the main result from [Bo–Ta] in two aspects: to examine all possible values for the κ -entropy and “get rid” of possible pathological actions on which such values are attained. In fact, we show more.

Main Theorem. *Let G do not have property (T). Let κ be a generating measure on G . Then the following are satisfied.*

- (1) *For each real $t \in (0, +\infty)$, there is a type III_1 ergodic free nonsingular action $T = (T_g)_{g \in G}$ on a standard probability space (X, μ) such that $h_\kappa(T, \mu) = t$.*
- (2) *For each real $t \in (0, +\infty)$, there is $\lambda \in (0, 1)$ and a type III_λ ergodic free nonsingular action $T = (T_g)_{g \in G}$ on a standard probability space (X, μ) such that $h_\kappa(T, \mu) = t$.*

The proof is based on the measurable orbit theory (see [FeMo], [Sc1] and a survey [DaSi]) and cohomology properties of non-strongly ergodic actions [Sc2], [JoSc].

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1. SOME BACKGROUND ON ORBIT THEORY

Let T be an ergodic free nonsingular action of G on a standard nonatomic probability space (X, \mathfrak{B}, μ) . Denote by \mathcal{R} the T -orbit equivalence relation on X . We recall that the *full group* $[\mathcal{R}]$ of \mathcal{R} consists of all one-to-one nonsingular transformations r of (X, μ) such that the graph of r is a subset of \mathcal{R} . Given a locally compact second countable group H , denote by λ_H a left Haar measure on H . A Borel map $\alpha : \mathcal{R} \rightarrow H$ is called a *cocycle* of \mathcal{R} if $\alpha(x, y) = \alpha(x, z)\alpha(z, y)$ for all points x, y, z from a μ -conull subset of X such that $x \sim_{\mathcal{R}} y \sim_{\mathcal{R}} z$. By $T(\alpha) = (T(\alpha)_g)_{g \in G}$ we denote the α -skew product extension of T , i.e. a G -action on the product space $(X \times H, \mu \times \lambda_H)$:

$$T(\alpha)_g(x, h) = (T_g x, \alpha(T_g x, x)h).$$

It is obvious that $T(\alpha)$ is $(\mu \times \lambda_H)$ -nonsingular. We say that α is *ergodic* if $T(\alpha)$ is ergodic. Consider the H -action on $(X \times H, \mu \times \lambda_H)$ by rotations (from the right) along the second coordinate. It commutes with $T(\alpha)$. The restriction of this action to the sub- σ -algebra of $T(\alpha)$ -invariant Borel subsets is called *the action of H associated with α* . It is ergodic. It is trivial if and only if α is ergodic. If $H = \mathbb{R}_+^*$ and $\alpha(T_g x, x) := \log \frac{d\mu \circ T_g}{d\mu}(x)$ at a.e. x for each $g \in G$ then α is called *the Radon-Nikodym cocycle* of \mathcal{R} . It does not depend on the choice of nonsingular group action generating \mathcal{R} . The corresponding associated action of \mathbb{R}_+^* is called *the associated flow* of T . If two group actions are orbit equivalent then their associated flows are isomorphic. The associated flow is transitive and free if and only if T admits an σ -finite invariant μ -equivalent measure. In this case T is said to be *of type II*. If the invariant measure is finite, T is said to be of type II_1 . If the invariant measure

is infinite, T is said to be of type II_∞ . If T does not admit an invariant equivalent measure then T is said to be of type III . Type III admits further classification into subtypes III_λ , $0 \leq \lambda \leq 1$. If the associated flow of T is periodic with period $-\log \lambda$ for some $\lambda \in (0, 1)$ then T is said to be of type III_λ . If the associated flow is trivial (on a singleton) then T is said to be of type III_1 . Equivalently, T is of type III_1 if and only if the Radon-Nikodym cocycle of T is ergodic. If T is of type III but not of type III_λ for any $\lambda \in (0, 1]$ then T is said to be of type III_0 .

Lemma 1.1. *Let μ be invariant under T . Let H be discrete and countable. Let $\alpha : \mathcal{R} \rightarrow H$ be an ergodic cocycle. Then the following holds.*

- (i) *The subrelation $\mathcal{R}_0 := \{(x, y) \in \mathcal{R} \mid \alpha(x, y) = 1\}$ of \mathcal{R} is ergodic.*
- (ii) *For each $h \in H$, there is an element $r_h \in [\mathcal{R}]$ such that $\alpha(r_h x, x) = h$ for μ -a.e. $x \in X$.*

Idea of the proof. Pass to the ergodic skew product extension $T(\alpha)$ and use the following Hopf lemma: if $D = (D_h)_{h \in H}$ is an ergodic H -action of type II and λ is a D -invariant equivalent measure then for all subsets A and B with $\lambda(A) = \lambda(B)$, there is a Borel bijection $\tau : A \rightarrow B$ such that the graph of τ is a subset of the D -orbit equivalence relation. \square

Let S be a nonsingular H -action on a standard probability space (Y, \mathfrak{Y}, ν) . Given a cocycle $\alpha : \mathcal{R} \rightarrow H$, we can form a *skew product action* $T(\alpha, S) = (T(\alpha, S)_g)_{g \in G}$ of G on the product space $(X \times Y, \mu \times \nu)$ by setting

$$T(\alpha, S)_g(x, y) := (T_g x, S_{\alpha(T_g x, x)} y).$$

Then $T(\alpha, S)$ is $(\mu \times \nu)$ -nonsingular.

Lemma 1.2. *Let T, μ, H be as in Lemma 1.1. If α is ergodic and S is ergodic then $T(\alpha, S)$ is also ergodic. The associated flow of $T(\alpha, S)$ is isomorphic to the associated flow of S . In particular, the type of $T(\alpha, S)$ equals the type of S .*

Proof. Let $F : X \times Y \rightarrow \mathbb{R}$ be a Borel function. If F is $T(\alpha, S)$ -invariant then $F(x, y) = F(x', y)$ if $(x', x) \in \mathcal{R}_0$ for a.e. y . By Lemma 1.1(i), $F(x, y) = f(y)$ for some Borel function $f : Y \rightarrow \mathbb{R}$. Lemma 1.1(ii) now yields that f is invariant under S . Since S is ergodic, f is constant mod ν . Thus F is constant mod $\mu \times \nu$. Hence $T(\alpha, S)$ is ergodic.

The second claim of the lemma follows from Lemma 1.1, the fact that \mathcal{R} is generated by \mathcal{R}_0 and the family of transformations $(r_h)_{h \in H}$ and that $r_h[\mathcal{R}_0]r_h^{-1} = [\mathcal{R}_0]$ for each $h \in H$. \square

We now recall the definition of strongly ergodic actions (see [CoWe], [JoSc] and references therein). Let T be an ergodic nonsingular G -action on non-atomic probability space (X, \mathfrak{B}, μ) . A sequence $(B_n)_{n \in \mathbb{N}}$ in \mathfrak{B} is called *asymptotically invariant* if $\lim_{n \rightarrow \infty} \mu(B_n \Delta T_g B_n) = 0$ for every $g \in G$. If every asymptotically invariant sequence $(B_n)_{n \in \mathbb{N}}$ is trivial, i.e. $\lim_{n \rightarrow \infty} \mu(B_n)(1 - \mu(B_n)) = 0$ then T is called *strongly ergodic*. We note the the strong ergodicity is invariant under the orbit equivalence. We will need the following lemma.

Lemma 1.3.

- (i) *If G does not have property (T) then there is an ergodic probability preserving free action T of G which is not strongly ergodic [CoWe].*

- (ii) If T is an ergodic nonsingular free action of G which is not strongly ergodic then for each countable discrete Abelian group A , there is an ergodic cocycle of the T -orbit equivalence relation with values in A (see [Sc, Corollary 1.5] and Theorem A2 below¹).

2. PROOF OF THE MAIN RESULT

The following lemma is almost a literal repetition of [Bo–Ta, Lemma 4.1], where it was proved under an additional assumption that T is measure preserving.

Lemma 2.1 (Entropy addition formula). *Let κ be a probability on G and let T be a nonsingular action of G on a standard probability space (X, \mathfrak{B}, μ) . Given a discrete countable group H and a nonsingular action $S = (S_h)_{h \in H}$ of H on a standard probability space (Y, \mathfrak{F}, ν) , let κ_x denote the pushforward of κ under the map $G \ni g \mapsto \alpha(T_g x, x) \in H$ for each $x \in X$. Then*

$$h_\kappa(T(\alpha), \mu \times \nu) = h_\kappa(T, \mu) + \int_X h_{\kappa_x}(S, \nu) d\mu(x).$$

Proof.

$$\begin{aligned} h_\kappa(T(\alpha, S), \mu \times \nu) &= - \sum_{g \in G} \kappa(g) \int_{X \times Y} \log \left(\frac{d(\mu \times \nu) \circ T_g(\alpha)}{d(\mu \times \nu)}(x, y) \right) d\mu(x) d\nu(y) \\ &= h_\kappa(T, \mu) - \int_X \sum_{g \in G} \kappa(g) \int_Y \log \left(\frac{d\nu \circ S_{\alpha(T_g x, x)}}{d\nu}(y) \right) d\nu(y) d\mu(x) \\ &= h_\kappa(T, \mu) - \int_X \sum_{h \in H} \kappa_x(h) \int_Y \log \left(\frac{d\nu \circ S_h}{d\nu}(y) \right) d\nu(y) d\mu(x) \\ &= h_\kappa(T, \mu) + \int_X h_{\kappa_x}(S, \nu) d\mu(x). \quad \square \end{aligned}$$

Proof of Main Theorem. We will proceed in two steps. On the first step, for each $\epsilon > 0$, we construct an ergodic nonsingular G -action of type III_1 (or of type III_λ for some $\lambda \in (0, 1)$) whose κ -entropy is less than ϵ . On the second step we show how to change the quasiinvariant measure for the action constructed on the first step with appropriate equivalent measures to force the κ -entropy to attain all the values from the interval $(\epsilon, +\infty)$.

Step 1. Fix $\epsilon > 0$ and $\lambda \in (0, 1)$. Fix an enumeration $G = \{g_n \mid n \in \mathbb{N}\}$ and a sequence of integers $1 = l_1 \leq l_2 \leq \dots$ such that $l_{n+1} - l_n \leq 1$ for all $n \in \mathbb{N}$, $l_n \rightarrow \infty$ and

$$(2-1) \quad \sum_{n=1}^{\infty} \kappa(g_n)(l_n + 1) < 2.$$

By Lemma 1.3(i), there is a measure preserving free action T of G on a standard probability space (X, \mathfrak{B}, μ) which is not strongly ergodic. Denote by \mathcal{R} the T -orbit

¹Since the proof of [Sc, Corollary 1.5] was not completed there we provide a complete proof of it in Appendix A.

equivalence relation. Let $F := \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. We consider the elements of F as $\mathbb{Z}/2\mathbb{Z}$ -valued functions on \mathbb{N} with finite support. Given $f \in F$, we let

$$\|f\| := \max\{j \in \mathbb{N} \mid f(j) \neq 0\}.$$

By Lemma 1.3(ii), there exists an ergodic cocycle $\alpha : \mathcal{R} \rightarrow F$. For each $n \in \mathbb{N}$, we can choose $M_n \in \mathbb{N}$ such that

$$(2-2) \quad \mu\left(\left\{x \in X \mid \max_{1 \leq i \leq n} \|\alpha(T_{g_i}x, x)\| < M_n\right\}\right) > 1 - \frac{1}{n2^n}.$$

Without loss of generality we may assume that $M_n = M_{n+1}$ if and only if $l_n = l_{n+1}$ for each $n \in \mathbb{N}$. Let $N := \{f \in F \mid f(M_n) = 0 \text{ for each } n \in \mathbb{N}\}$. Then N is a subgroup of F . The quotient group F/N is identified naturally with the “complimentary to F ” subgroup $\{f \in F \mid f(n) = 0 \text{ for each } n \neq M_1, M_2, \dots\}$ which is, in turn, isomorphic to F in a natural way. Hence passing from α to the quotient cocycle

$$\alpha + N : \mathcal{R} \ni (x, y) \mapsto \alpha(x, y) + N \in F/N$$

means that we may assume without loss of generality that $M_n = l_n$ for each $n \in \mathbb{N}$ in (2-2). (We use here a simple fact that $\alpha + N$ is ergodic whenever α is.) Therefore applying (2-2) we obtain that

$$(2-3) \quad \begin{aligned} \int_X \|\alpha(T_{g_n}x, x)\| d\mu(x) &= \sum_{s=1}^{\infty} s\mu(\{x \in X \mid \|\alpha(T_{g_n}x, x)\| = s\}) \\ &\leq l_n + \sum_{s>l_n} s\mu(\{x \in X \mid \|\alpha(T_{g_n}x, x)\| = s\}) \\ &\leq l_n + \sum_{s>l_n} \frac{1}{2^s} \\ &\leq l_n + 1. \end{aligned}$$

The second inequality here follows from the fact that for each $s > l_n$, we have $s = l_m$ for some $m \geq n$ and hence

$$\mu(\{x \in X \mid \|\alpha(T_{g_n}x, x)\| = s\}) \leq \frac{1}{m2^m} \leq \frac{1}{s2^s}.$$

because $m \geq l_m = s$.

Now we consider F as a (dense) subgroup of the compact Abelian group $K := (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ of all $\mathbb{Z}/2\mathbb{Z}$ -valued functions on \mathbb{N} . Denote by S the action of H on K by translations. Let ν_n denote the distribution on $\mathbb{Z}/2\mathbb{Z}$ such that

$$\nu_n(0) = \frac{1}{1 + e^{\epsilon_n}}, \quad \nu_n(1) = \frac{e^{\epsilon_n}}{1 + e^{\epsilon_n}}$$

for some sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of reals such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$, $\sum_{n \in \mathbb{N}} \epsilon_n^2 = \infty$ and $\max\{|\epsilon_n| \mid n \in \mathbb{N}\} < \epsilon$. Let $\nu = \bigotimes_{n \in \mathbb{N}} \nu_n$. Then S is ν -nonsingular, ergodic and of type III_1 [ArWo]. Therefore by Lemma 1.2, the skew product G -action $T(\alpha, S)$ on

$(X \times K, \mu \times \nu)$ is ergodic and of type III_1 . Hence $h_\kappa(T(\alpha, S)) > 0$. To estimate $h_\kappa(T(\alpha, S))$ from above, we first let $N_f := \{n \in \mathbb{N} \mid f(n) \neq 0\}$ for $f \in F$. It is obvious that $\#N_f \leq \|f\|$. Since

$$\begin{aligned} - \int_Y \log \left(\frac{d\nu \circ S_f}{d\nu} \right) (y) d\nu(y) &= - \sum_{n \in N_f} \int_{\mathbb{Z}/2\mathbb{Z}} \log \left(\frac{\nu_n(y_n + 1)}{\nu_n(y_n)} \right) d\nu_n(y_n) \\ &= \sum_{n \in N_f} (\nu_n(1) - \nu_n(0)) \log \frac{\nu_n(1)}{\nu_n(0)}, \end{aligned}$$

we obtain that for each probability ξ on F ,

$$\begin{aligned} h_\xi(S, \nu) &= \sum_{f \in F} \xi(f) \sum_{n \in N_f} (\nu_n(1) - \nu_n(0)) \log \frac{\nu_n(1)}{\nu_n(0)} \\ (2-4) \quad &\leq \sum_{f \in F} \xi(f) \sum_{n \in N_f} |\epsilon_n| \\ &\leq \epsilon \|f\| \sum_{f \in F} \xi(f). \end{aligned}$$

Since T preserves κ , it follows that $h_\kappa(T, \mu) = 0$. Then Lemma 2.1, (2-4) and (2-3) yield that

$$\begin{aligned} h_\kappa(T(\alpha, S), \mu \times \nu) &\leq \epsilon \int_X \sum_{f \in F} \kappa_x(f) \|f\| d\mu(x) \\ &= \epsilon \sum_{g \in G} \kappa(g) \int_X \|\alpha(T_g x, x)\| d\mu(x) \\ &\leq \epsilon \sum_{n=1}^{\infty} \kappa(g_n) (l_n + 1). \end{aligned}$$

It now follows from (2-1) that $h_\kappa(T(\alpha, S), \mu \times \nu) \leq 2\epsilon$. Hence

$$\inf\{h_\kappa(A) \mid A \text{ is a type } III_1 \text{ ergodic free action of } G\} = 0.$$

In a similar way we may show that

$$(2-5) \quad \inf\{h_\kappa(A) \mid A \text{ is a type } III_\lambda \text{ ergodic free action of } G, \lambda \in (0, 1)\} = 0.$$

For that we argue as above but with a different measure ν . Indeed, let ν_n denote the distribution on $\mathbb{Z}/2\mathbb{Z}$ such that

$$\nu_n(0) = \frac{1}{1 + e^\epsilon}, \quad \nu_n(1) = \frac{e^\epsilon}{1 + e^\epsilon}$$

Let $\nu = \bigotimes_{n \in \mathbb{N}} \nu_n$. Then S is ν -nonsingular, ergodic and of type $III_{e^{-\epsilon}}$ [ArWo]. Therefore by Lemma 1.2, the skew product G -action $T(\alpha, S)$ on $(X \times K, \mu \times \nu)$ is ergodic and of type $III_{e^{-\epsilon}}$. Hence $h_\kappa(T(\alpha, S)) > 0$. As in the III_1 -case considered above, we obtain that $h_\kappa(T(\alpha, S), \mu \times \nu) < 2\epsilon$ and hence (2-5) follows.

Step 2. Given $\epsilon > 0$, let ν be a measure on K such that

$$(2-5) \quad h_\kappa(T(\alpha, S), \mu \times \nu) < \epsilon.$$

We choose $n_0 > 0$ such that

$$(2-6) \quad \int_X \kappa_x(\{f \in F \mid f(n_0) \neq 0\}) d\mu(x) > 0.$$

It exists because otherwise we would have that κ_x is supported at 0 for a.e. $x \in X$. The latter yields that $\alpha(T_g x, x) = 0$ at a.e. x for all g from the support of κ . Since κ is generating, it follows that α is trivial, a contradiction.

Let ω be a probability on $\mathbb{Z}/2\mathbb{Z}$ supported at 1^2 . For each $\theta \in (0, 1]$, we let

$$\nu_n^\theta = \begin{cases} \nu_n, & \text{if } n \neq n_0 \\ \theta \nu_n + (1 - \theta)\omega, & \text{if } n = n_0 \end{cases}$$

and $\nu^\theta := \bigotimes_{n \in \mathbb{N}} \nu_n^\theta$. Then ν^θ is equivalent to ν and hence $\mu \times \nu^\theta$ is equivalent to $\mu \times \nu$. Therefore the dynamical systems $(T(\alpha, S), \mu \times \nu^\theta)$ and $(T(\alpha, S), \mu \times \nu)$ are of the same Krieger's type. It follows from the equality in (2-4) that

$$h_{\kappa_x}(S, \nu) - h_{\kappa_x}(S, \nu^\theta) = \kappa_x(\{f \in F \mid f(n_0) \neq 0\})(\Phi(\nu_{n_0}(0)) - \Phi(\nu_{n_0}^\theta(0))),$$

where $\Phi(t) := (1 - 2t) \log \frac{1-t}{t}$, if $t \in (0, 1)$. Therefore the map

$$(0, 1] \ni \theta \mapsto h_\kappa(T(\alpha, S), \mu \times \nu^\theta) = \int_X h_{\kappa_x}(S, \nu^\theta) d\mu(x) \in \mathbb{R}$$

is continuous. In view of (2-6), this map goes to infinity as $\theta \rightarrow 0$. Since $\nu^1 = \nu$ and (2-5) holds, it follows that

$$\{h_\kappa(T(\alpha, S), \mu \times \nu^\theta) \mid \theta \in (0, 1]\} \supset (\epsilon, \infty),$$

as desired. \square

APPENDIX A

Let \mathcal{R} be an ergodic measure preserving countable equivalence relation on a nonatomic standard probability space (X, \mathfrak{B}, μ) and let G be a locally compact second countable group. A cocycle $\rho : \mathcal{R} \rightarrow G$ is called *regular* if the action of G associated with ρ is transitive. For instance, an ergodic cocycle is regular. A coboundary is also regular.

Proposition A1. *Let A be an amenable discrete countable group and let H be a locally compact second countable amenable group. Let $\alpha : \mathcal{R} \rightarrow A$ be a cocycle. If α is not regular then there is an ergodic cocycle of \mathcal{R} with values in H .*

Proof. Let \mathcal{A} stand for the “transitive” equivalence relation on A , i.e. $\mathcal{A} = A \times A$. Let λ be a probability measure on A which is equivalent to Haar measure. Then the equivalence relation $\mathcal{R} \times \mathcal{A}$ is an ergodic equivalence relation on the probability

²We consider the group $\mathbb{Z}/2\mathbb{Z}$ as $\{0, 1\}$ with addition mod 2.

space $(X \times A, \mu \times \lambda)$. Let $V = (V_a)_{a \in A}$ denote the nonsingular action of A on $(X \times A, \mu \times \lambda)$ by right rotations along the second coordinate. Then $\mathcal{R} \times \mathcal{A}$ is generated by a subrelation $\mathcal{R}(\alpha)$ and V , i.e. two points $z_1, z_2 \in X \times A$ are $(\mathcal{R} \times \mathcal{A})$ -equivalent if and only if the points $V_{a_1}z_1$ and $V_{a_2}z_2$ are $\mathcal{R}(\alpha)$ -equivalent for some $a_1, a_2 \in A$. Let $W = (W_a)_{a \in A}$ stand for the action of A associated with α . Denote by (Ω, ν) the space of this action. Then we can assume that there is a Borel map $\pi : X \times A \rightarrow \Omega$ such that $\nu = (\mu \times \lambda) \circ \pi^{-1}$ and $\pi \circ V_a = W_a \circ \pi$ for each $a \in A$. We observe that π is the $\mathcal{R}(\alpha)$ -ergodic decomposition of $X \times A$. Since A is amenable, the W -orbit equivalence relation \mathcal{I} on (Ω, ν) is hyperfinite [Co–We]. It is ergodic. Since α is not regular, \mathcal{I} is non-transitive. Hence there is an ergodic cocycle $\beta : \mathcal{I} \rightarrow H$ (see [He] and [GoSi]). We now define a cocycle $\beta^* : \mathcal{R} \times \mathcal{A} \rightarrow H$ by setting

$$\beta^*(z_1, z_2) := \beta(\pi(z_1), \pi(z_2)).$$

Then β^* is well defined. Since π is the $\mathcal{R}(\alpha)$ -ergodic decomposition, it follows that β^* is ergodic. Restricting $\mathcal{R} \times \mathcal{A}$ and β^* to the subset $X \times \{1_A\}$ we obtain \mathcal{R} and a cocycle of \mathcal{R} with values in H respectively. Of course, this cocycle is also ergodic. \square

Let A be an Abelian locally compact noncompact second countable group. For a cocycle $\alpha : \mathcal{R} \rightarrow A$, we denote by $E(\alpha) \subset A \sqcup \{\infty\}$ the essential range of α (see [Sc1, Definition 3.1]). The following theorem provides a complete proof of [Sc2, Corollary 1.5] (it was assumed additionally in [Sc2] that A is Abelian).

Theorem A2. *Let $T = (T_g)_{g \in G}$ be an ergodic measure preserving action of G on a standard nonatomic probability space (X, \mathfrak{B}, μ) . Let A be a countable amenable group. If T is not strongly ergodic then there is an ergodic cocycle of the T -orbit equivalence relation \mathcal{R} with values in A .*

Proof. Let \mathcal{R} denote the T -orbit equivalence relation. Let K, F, S, N_f and $\|\cdot\|$ denote the same objects as in the proof of Main Theorem. Let λ_K stand for the Haar measure on K . For each $n \in \mathbb{N}$, denote by f_n the element of F such that $N_{f_n} = \{n\}$. By [JoSc, Lemma 2.4], there are a Borel map $\pi : X \rightarrow K$ and a sequence $(V_n)_{n \in \mathbb{N}}$ of transformations in $[\mathcal{R}]$ such that $\mu \circ \pi^{-1} = \lambda_K$,

$$\{\pi(T_g x) \mid g \in G\} = \{S_f \pi(x) \mid f \in F\}$$

and $\pi(V_n x) = S_{f_n} \pi(x)$ for a.a. $x \in X$. Denote by \mathcal{S} the S -orbit equivalence relation on K . We define a cocycle $\beta : \mathcal{S} \rightarrow F$ by setting $\beta(S_f y, y) := f, y \in Y, f \in F$. By [Sc1, Proposition 3.15], $E(\beta) = \{0, +\infty\}$. We now define a cocycle $\beta^* : \mathcal{R} \rightarrow F$ by setting

$$\beta^*(T_g x, x) := \beta(\pi(T_g x), \pi(x)),$$

$x \in X, g \in G$. Since $E(\beta^*) \subset E(\beta)$, we obtain that either $E(\beta^*) = \{0, \infty\}$ or $E(\beta^*) = \{0\}$. In the latter case, β^* is a coboundary [Sc1]. Hence there is a Borel map $\xi : X \rightarrow K$ such that $\beta^*(V_n x, x) = \xi(V_n x) - \xi(x)$ for each n at a.e. $x \in X$. There is a constant $C > 0$ and a subset $X_C \subset X$ such that $\mu(X_C) > 3/4$ and $\|\xi(x)\| < C$ for all $x \in X_C$. It follows that for each n ,

$$\sup_{x \in X_C \cap V_n^{-1} X_C} \|\beta^*(V_n x, x)\| < C.$$

This contradicts to the fact that $\|\beta^*(V_n x, x)\| = n$ for a.a. $x \in X$ and $n \in \mathbb{N}$. Hence $E(\beta^*) = \{0, \infty\}$. This yields that β^* is not regular. It remains to apply Proposition A1. \square

REFERENCES

- [ArWo] H. Araki and E. J. Woods, *A classification of factors*, Pub. RIMS Kyoto Univ., Ser. A **3** (1968), 51–130.
- [Be–Va] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan’s property (T)*, Cambridge University Press, 2008.
- [Bo] L. Bowen, *Random walks on coset spaces with applications to Furstenberg entropy*, Invent. Math. **196** (2014), 485–510.
- [Bo–Ta] L. Bowen, Y. Hartman and O. Tamuz, *Property (T) and the Furstenberg entropy of nonsingular actions*, Proc. Amer. Math. Soc. **144** (2016), 31–39.
- [CoWe] A. Connes and B. Weiss, *Property T and asymptotically invariant sequences*, Israel J. Math. **37** (1980), 209–210.
- [Co–We] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory and Dynamical Systems **1** (1981), 431–450.
- [CoWo] A. Connes and E. J. Woods, *Approximately transitive flows and ITPFI factors*, Ergodic Theory Dynamical Systems **5** (1985), 203–236.
- [DaSi] A. I. Danilenko and C. E. Silva, *Ergodic theory: nonsingular transformations*, Encyclopedia of Complexity and Systems Science, Springer, 2009, pp. 3055–3083.
- [FeMo] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. **234** (1977), 289–324.
- [Fu] H. Furstenberg, *Noncommuting random products*, Trans. Amer. Math. Soc. **108** (1963), 377–428.
- [GoSi] V. Ya. Golodets and S. D. Sinelshchikov, *Amenable ergodic group actions and images of cocycles*, Dokl. Akad. Nauk SSSR **312** (1990), 1296–1299 (Russian); Soviet Math. Dokl. **41** (1990), 523–526.
- [JoSc] V. F. R. Jones and K. Schmidt, *Asymptotically invariant sequences and approximate finiteness*, American Journal of Mathematics **109** (1987), 91–114.
- [HaTa] Y. Hartman and O. Tamuz, *Furstenberg entropy realizations for virtually free groups and lamplighter groups*, Journal d’Analyse Mathématique **126** (2015), 227–257.
- [He] M. Herman, *Construction de difféomorphismes ergodiques*, preprint (1979).
- [Ne] A. Nevo, *The spectral theory of amenable actions and invariants of discrete groups*, Geom. Dedicata **100** (2003), 187–218.
- [NeZi] A. Nevo and R. J. Zimmer, *Rigidity of Furstenberg entropy for semisimple Lie group actions*, Annales Scientifiques de l’Ecole Normale Supérieure **33** (2000), 321–343.
- [Sc1] K. Schmidt, *Cocycles of ergodic transformation groups*, Lecture Notes in Mathematics, Vol. 1, MacMillan (India), 1977.
- [Sc2] ———, *Cohomology and the absence of strong ergodicity for ergodic group actions*, Lecture Notes in Math., vol. 1132, Springer-Verlag, Berlin and New York, 1985, pp. 486–496.

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